Dynamic Analysis of Plate under Moving Load on Dynamic Foundation

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Abstract. This paper presents the analysis of the dynamic behavior of the thin plate on the dynamic foundation subjected to moving loads. The stiffness of the foundation is considered variable. A numerical survey using the finite element method is applied to analyze the time-dependent dynamic equation of the plate. The numerical results describe the weakened foundation cases in practice and compare the difference between the dynamic and viscoelastic foundations.

Keywords: dynamic foundation, variable stiffness foundation, thin plate, moving load, finite element method.

1 Introduction

The behavior of the plate on the foundation is a fundamental problem, widely studied and applied, especially in infrastructure engineering. Modeling of variable stiffness foundation under moving vehicle loads is a common problem in practice. When the road is under operation, the foundation has a great change, leading to the structure being easily damaged, causing unsafety in the process of movement, and loss of comfort during vehicle control. This usually happens in places where surface water levels often rise for a long time, the ground is soft soil, the design is not suitable, or the construction is not met requirements.

There are some articles that investigated the structures on elastic foundations with variable stiffness, typically recent studies such as: Free vibration analysis of beams on variable Winkler elastic foundation (constant, linear, and second order) by using the differential transform method [1]; Dynamic response to moving masses of rectangular plates with different boundary conditions and resting on variable Winkler foundation (change of value of the stiffness in each particular case) [2,3]; Analytical solution for the elastic bending of beams lying on a variable Winkler support (an inverse of a fourth order polynomial) [4]; An analytical solution for free vibration of elastically restrained Timoshenko beam on an arbitrary variable Winkler foundation and under axial load [5]; Analytical solution for the elastic bending of beams lying on a linearly variable Winkler support [6]; Dynamic response of plates resting on a fractional viscoelastic foundation and subjected to a moving load [7]; Vibration of orthotropic rectangular plates under the action of moving distributed masses and resting on a variable elastic Pasternak foundation with clamped end conditions [8]; Dynamic analysis of railway track on variable foundation under harmonic moving load [9]; Dynamic response of railway track resting on variable foundation using finite element method [10]; Development of an analytical method for calculating beams on a variable elastic Winkler foundation [11].

However, the above studies have not clearly described the relationship between plate displacement and foundation stiffness when the stiffness changes and weakens. This paper will fully address the relationship to get an overall picture of the plate on an elastic foundation with variable stiffness subjected to moving loads. The elastic foundation is considered the dynamical foundation, a new foundation model developed recently [12–17]. The obtained numerical results will be compared with the viscoelastic foundation. From there, engineers have suitable options to strengthen, repair, and renovate the road sections with heterogeneous stiffness, which are weakened during the exploitation process due to subjective and objective causes.

2 Formulation

From the classic plate theory, the partial differential equation for the deflection of a plate on a viscoelastic foundation subjected to a moving load as

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + k_f(x)w + \left(c_p + c_f\right)\frac{\partial w}{\partial t} + (\rho h + m_f)\frac{\partial^2 w}{\partial t^2} = F(x, y, t) \tag{1}$$

where w is the plate deflection in the direction of the normal which is satisfied the above (1); D is the plate stiffness; E is Young's modulus; μ is Poisson's ratio; k_f is the foundation stiffness; c_p and c_f are damping constants of the plate and the foundation; ρ and h are the density and the thickness of the plate; m_f is the lumped mass of the foundation; t is the time variable; x and y are rectangular Cartesian coordinates in the plane of the plate; $F(x, y, t)$ is the loads and forces with (x, y) coordinates with respect to the time variable t acting on the plate resting on the dynamic foundation.

With the viscoelastic foundation, m_f is equal to 0.

To solve (1), the finite element method is adopted. Based on the results of the study on the dynamic response of plates on elastic foundation to moving loads [1], the dynamic equation of the plate-foundation system is

$$([\boldsymbol{M_p}] + [\boldsymbol{M_f}])\ddot{\boldsymbol{d}} + ([\boldsymbol{C_p}] + [\boldsymbol{C_f}])\dot{\boldsymbol{d}} + ([\boldsymbol{K_p}] + [\boldsymbol{K_f^x}])\boldsymbol{d} = \boldsymbol{F} \tag{2}$$

Where $([M_p] + [M_f])$ are the mass matrices, $([C_p] + [C_f])$ are the damping matrices, and $([K_p] + [K_f^x])$ are the stiffness matrices of the plate and the foundation respectively (in this case $k_f(x)$ is considered as a function with respect to variable *x*); *F* is the load vector; *d* is the complete displacement vector.

From the theory of the finite element method for the plate bending element, let $[N]$ and $[B]$ be the shape function matrix and the gradient matrix, respectively, e is the plate bending element under consideration of minimum potential energy. The mass matrix $[M_f]$, stiffness matrix $\left[K_f^x\right]$, and damping matrix $[C_f]$ of the foundation are obtained.

$$\left[\boldsymbol{M_f}\right] = \int_e [\boldsymbol{N}]^{\boldsymbol{T}} \boldsymbol{m_f} [\boldsymbol{N}] \boldsymbol{dxdy} \tag{3}$$

$$\left[\boldsymbol{K_f^x}\right] = \int_e [\boldsymbol{N}]^{\boldsymbol{T}} \boldsymbol{k_f}(\boldsymbol{x}) [\boldsymbol{N}] \boldsymbol{dxdy} \tag{4}$$

$$\left[\boldsymbol{C_f}\right] = \int_e [\boldsymbol{N}]^{\boldsymbol{T}} \boldsymbol{c_f} [\boldsymbol{N}] \boldsymbol{dxdy} \tag{5}$$

The mass matrix $[M_p]$, the plate stiffness matrix $[K_p]$, the plate damping matrix $[C_p]$ of the plate are taken according to the above study [18], specifically the mass matrix and the stiffness matrix are given by

$$\left[\boldsymbol{M_p}\right] = \int_e [\boldsymbol{N}]^{\boldsymbol{T}} \boldsymbol{\rho h} [\boldsymbol{N}] \boldsymbol{dxdy} \tag{6}$$

$$\left[\boldsymbol{K_p}\right] = \int_e [\boldsymbol{B}]^{\boldsymbol{T}} [\boldsymbol{D}] [\boldsymbol{B}] \boldsymbol{dxdy} \tag{7}$$

The plate damping matrix is used Rayleigh damping as

$$\left[\boldsymbol{C_p}\right] = \boldsymbol{\alpha}\left[\boldsymbol{M_p}\right] + \boldsymbol{\beta}\left[\boldsymbol{K_p}\right] \tag{8}$$

with α and β are determined by the first two modes through modal analysis. And the load vector is

$$\left[\boldsymbol{C_p}\right] = \boldsymbol{\alpha}\left[\boldsymbol{M_p}\right] + \boldsymbol{\beta}\left[\boldsymbol{K_p}\right] \tag{9}$$

From the above analysis steps, equation (2) can be solved by Newmark's method. The problem can be considered with different boundary conditions.

3 Numerical Investigation

To implement numerical investigation and analysis, a computer program written in Python programming language [19] with open-source libraries including numpy [20], matplotlib [21], and openseespy [22] is used.

3.1 Verification of the written computer program

The numerical example from the program was compared with the study of M.-H. Huang and D. P. Thambiratnam [18] (see Fig. 1 and Fig. 2).

This is shown that the computer program using the finite element method and dynamic analysis step by step is reliable.

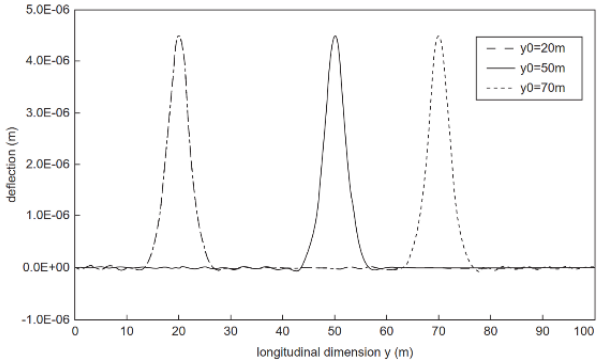

Fig. 1. The results of [18]

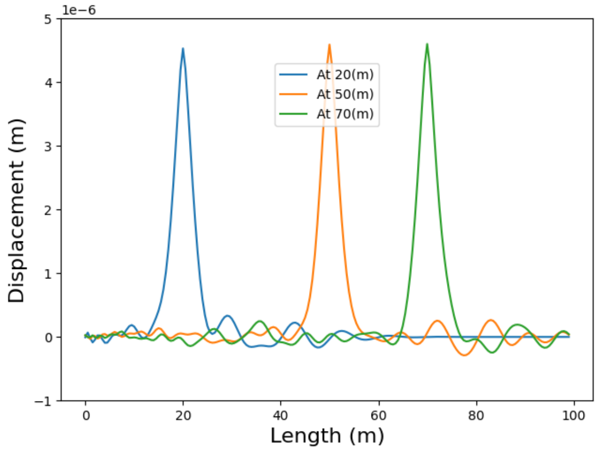

Fig. 2. The written computer program

3.2 Numerical Analysis

A simply supported LxB=100(m)x10(m) thin plate with thickness thk=0.3(m) resting on the dynamic foundation. The data for the plate and the load amplitude are given by: E=3.1x10^{10}(N/m^2), μ=0.25, ρ=2,440(kg/m^3), damping ratio ξ=5(%), K_f=10^7(N/m^3), C_f=100(Ns/m^3), m_f=1800(kg/m^3), P_z=100(kN).

The load moves along the center line and parallel to the long edge of the plate. It moves into the plate with an initial velocity of v=20(m/s) and this velocity remains constant. The direction of the load is perpendicular to the plate, and towards the plane of the plate.

The foundation stiffness is analyzed with the following 7 cases:

1) The foundation stiffness is constant.

2) The foundation stiffness is constant but at positions $L/4$, $L/2$ and $3L/4$ have $K = 0$ with a length of 2m (the foundation is depressed).

3) The foundation stiffness decreases in steps from position 0 to $L/2$ and increases symmetrically from $L/2$ to L, at $L/2$ has $K = 0$ with a length of 2m (the foundation is depressed).

4) The foundation stiffness decreases linearly from position 0 to $L/2$ and increases symmetrically from $L/2$ to L, at $L/2$ has $K = 0$ with a length of 2m (the foundation is depressed).

5) The foundation stiffness has a decrease in polynomial of order 2 from position 0 to $L/2$ and increases symmetrically from $L/2$ to L, at $L/2$ has $K = 0$ with a length of 2m (the foundation is depressed).

6) The foundation has a stiffness $K = 0$ (depressed at the joint) increasing in polynomial of order 2 from position 0 to $L/2$ and symmetrically decreasing from $L/2$ to L, at $L/2$ with a length of 2m, the foundation stiffness reaches the given K value.

7) The foundation has a stiffness $K = 0$ (depressed at the joint) and follows the law of sine function.

The plate is divided with 100 elements in the longitudinal direction, and 10 elements in the horizontal direction (N=1,111 nodes) with a ratio of two-element sides of 1:1 to achieve the best results. Then the stiffness of each spring is $k=K_f$xLxB/N=9,000,900.1(N/m), and the damping coefficient is $c=C_f$xLxB/N=90.01(Ns/m), the mass of each spring is m_fxLxBxh_f/N= 1,296.1296(kg) (h_f is the affection of mass foundation, in this problem h_f=80(cm)).

The problem is analyzed to investigate the displacements at positions $L/4$, $L/2$, $3L/4$, and the displacement spectrum of the center line parallel to the long side of the plate.

***Case 1**, the foundation stiffness is constant, $k_f(x) = k$:*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 3 and Fig. 4.

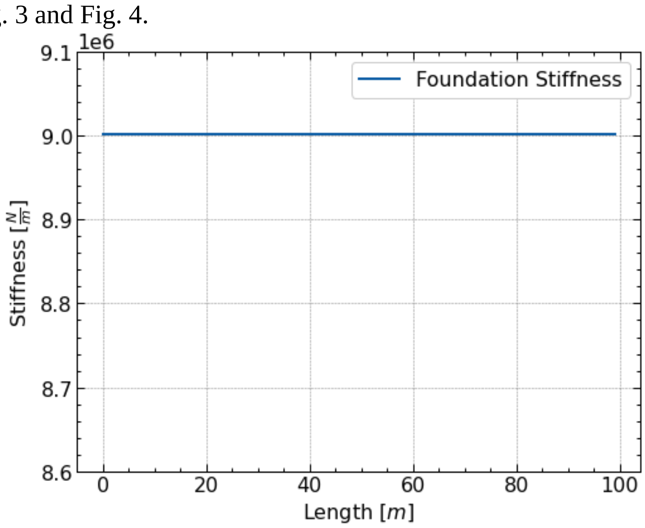

Fig. 3. Case 1 – Stiffness

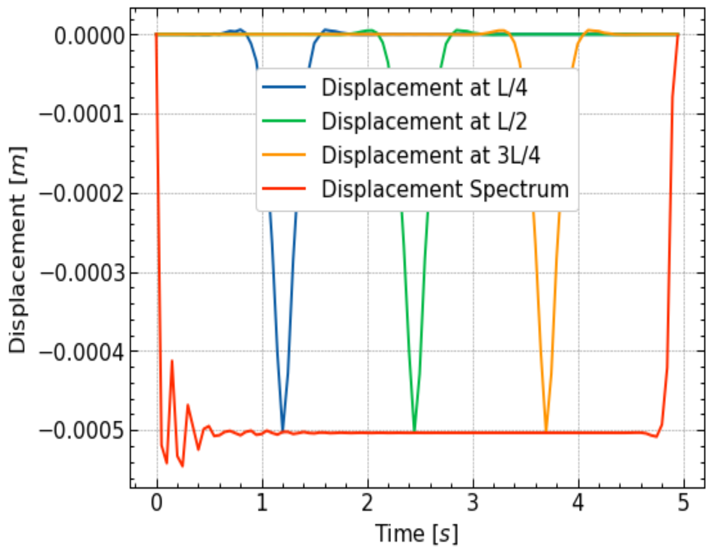

Fig. 4. Case 1 – Displacement

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 1.

Table 1. Case 1 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	-5.022.10^{-4} (m)	-5.033.10^{-4} (m)	-5.033.10^{-4} (m)
Viscoelastic	-5.013.10^{-4} (m)	-5.016.10^{-4} (m)	-5.016.10^{-4} (m)
Difference	0.18%	0.34%	0.34%

Remark: with the foundation having constant stiffness, the plate displaces relatively evenly under the effect of moving load. From the displacement spectrum with the initial speed of the load $v = 20(m/s)$, a non-zero acceleration at time $t = 0$ is provided for the first element of the plate under load, and gradually even under the resistance of the plate and the foundation. In practice, the transition position between the joint and the foundation is also a relatively easy position to damage.

Case 2*, the foundation stiffness is constant ($k_f(x) = k$) but at positions $L/4$, $L/2$ and $3L/4$ have $K = 0$ with a length of 2m (the foundation is depressed):*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 5 and Fig. 6.

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 2.

Remark: in case the plate is placed on a discontinuous foundation, as in practice, the locations have a sudden depression. These are the locations that are easy to cause damage to the plate if no timely and appropriate rectified measures are taken. From the displacement spectrum, we can see that at positions $L/4$, $L/2$ and $3L/4$, although the displacement increases significantly, the plate oscillation is relatively even because the plate elements in this position begin to oscillate when the load is moving close to.

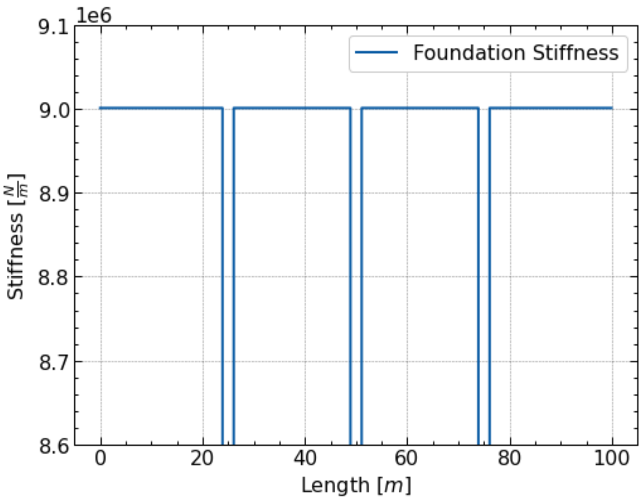

Fig. 5. Case 2 – Stiffness

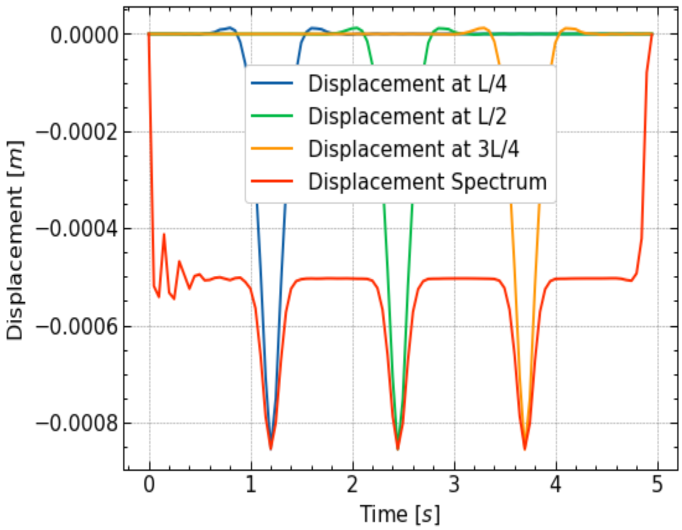

Fig. 6. Case 2 – Displacement

Table 2. Case 2 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	$-8.544.10^{-4}$ (m)	$-8.548.10^{-4}$ (m)	$-8.548.10^{-4}$ (m)
Viscoelastic	$-8.435.10^{-4}$ (m)	$-8.439.10^{-4}$ (m)	$-8.439.10^{-4}$ (m)
Difference	1.28%	1.28%	1.28%

Case 3*, the foundation stiffness decreases in steps from position 0 to* $L/2$ *and increases symmetrically from* $L/2$ *to* L*, at* $L/2$ *has* $K = 0$ *with a length of 2m (the foundation is depressed):*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 7 and Fig. 8.

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 3.

Table 3. Case 3 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	$-6.569.10^{-4}$ (m)	$-17.014.10^{-4}$ (m)	$-6.564.10^{-4}$ (m)
Viscoelastic	$-6.527.10^{-4}$ (m)	$-16.642.10^{-4}$ (m)	$-6.535.10^{-4}$ (m)
Difference	0.64%	2.19%	0.44%

Remark: the weakening of the foundation stiffness occurs quite often in practice due to the use of different materials. In this case, the stiffness is reduced to 0. From the displacement spectrum and the plate displacement plane, it shows that although the position $L/2$ has $K = 0$, the displacement is wider than in Case 2 above.

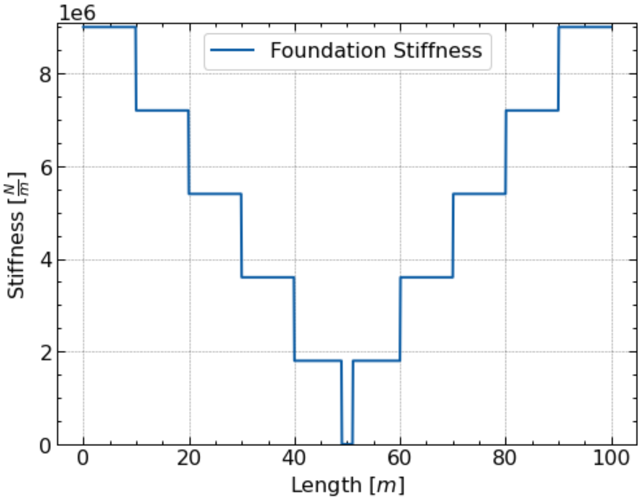

Fig. 7. Case 3 – Stiffness

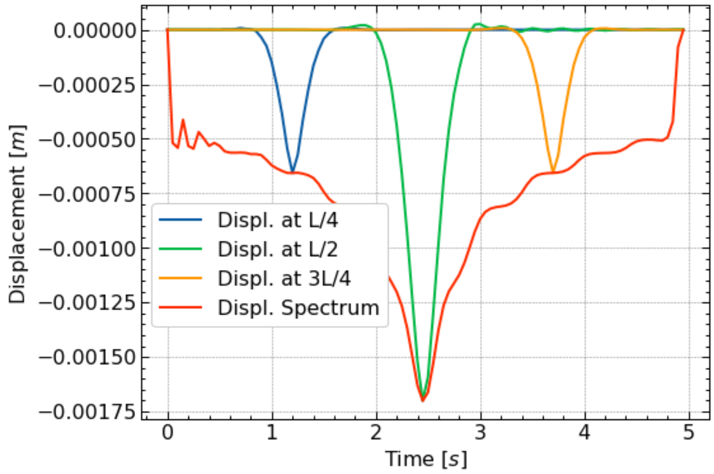

Fig. 8. Case 3 – Displacement

Case 4, *the foundation stiffness decreases linearly from position* 0 *to* $L/2$ $\left(k_f(x) = k - \frac{k}{\frac{49}{100}L}x\right)$ *and increases symmetrically from* $L/2$ *to* L $\left(k_f(x) = \frac{-51}{49}k + \frac{k}{\frac{49}{100}L}x\right)$*, at* $L/2$ *has* $K = 0$ *with a length of 2m (the foundation is depressed):*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 9 and Fig. 10.

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 4.

Table 4. Case 4 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	$-7.281.10^{-4}$ (m)	$-36.653.10^{-4}$ (m)	$-7.296.10^{-4}$ (m)
Viscoelastic	$-7.258.10^{-4}$ (m)	$-34.517.10^{-4}$ (m)	$-7.282.10^{-4}$ (m)
Difference	0.32%	5.83%	0.19%

Remark: This is the case of a weakening foundation described linearly, quite close to the phenomenon of water flooding in the foundation and seeping upwards to some extent. In this case, the analysis brings the weakened foundation down to a value of $K = 0$. From the displacement spectrum and the plate displacement plane, the displacement at the position with $K = 0$ is wider than that of Case 3 above.

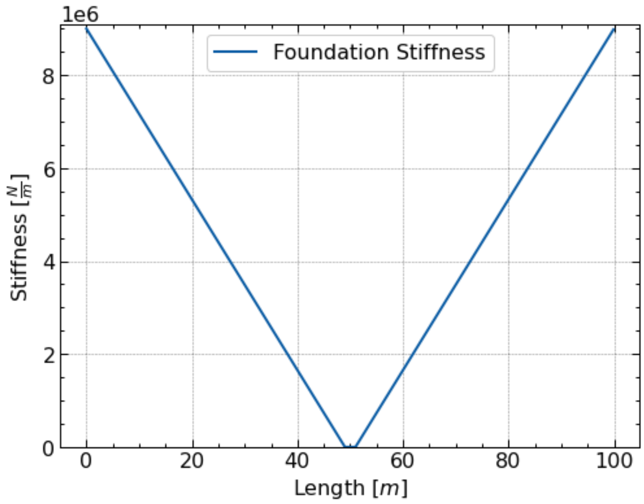

Fig. 9. Case 4 – Stiffness

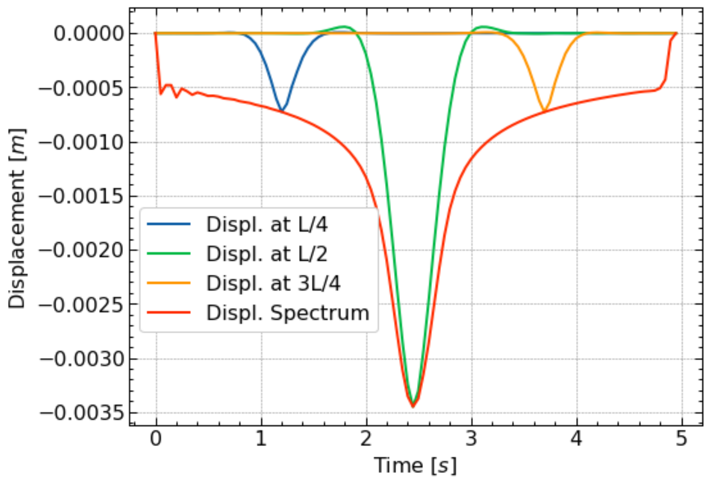

Fig. 10. Case 4 – Displacement

Case 5*, the foundation stiffness has a decrease in polynomial of order 2 from position 0 to L/2 (*$k_f(x) = k - \frac{9245}{2499}\frac{k}{L}x + \frac{9500}{2499}\frac{k}{L^2}x^2$*) and increases symmetrically from L/2 to L (*$k_f(x) = \frac{44}{49}k - \frac{9245}{2499}\frac{k}{L}x + \frac{9500}{2499}\frac{k}{L^2}x^2$*), at L/2 has* $K = 0$ *with a length of 2m (the foundation is depressed):*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 11 and Fig. 12.

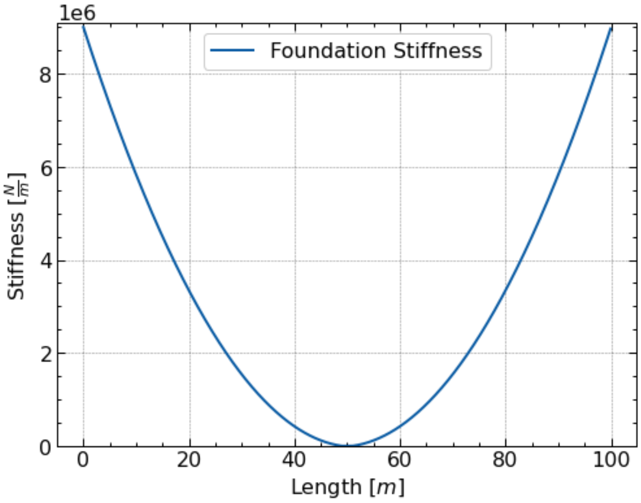

Fig. 11. Case 5 – Stiffness

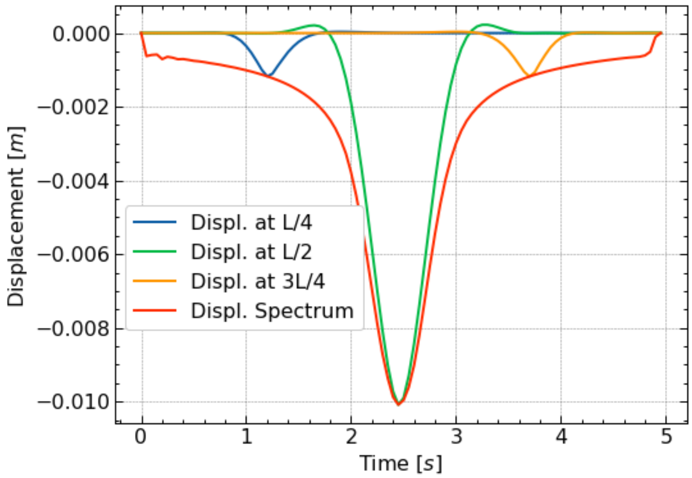

Fig. 12. Case 5 – Displacement

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 5.

Remark: like Case 4, but the weakening foundation is described as decreasing by a polynomial of order 2. From the displacement spectrum and the plate displacement plane, the displacement at $K = 0$ is wider compared to Case 4. This case causes the largest displacement at the center of the plate compared to the other cases.

Table 5. Case 5 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	-10.027.10^{-4} (m)	-101.001.10^{-4} (m)	-10.107.10^{-4} (m)
Viscoelastic	-10.035.10^{-4} (m)	-86.209.10^{-4} (m)	-10.090.10^{-4} (m)
Difference	0.08%	14.65%	0.17%

Case 6, *the foundation has a stiffness $K = 0$ (depressed at the joint) increasing in polynomial of order 2 from position 0 to L/2 ($k_f(x) = \frac{9755}{2499}\frac{k}{L}x - \frac{9500}{2499}\frac{k}{L^2}x^2$), and symmetrically decreasing from L/2 to L ($k_f(x) = \frac{5}{49}k + \frac{9245}{2499}\frac{k}{L}x - \frac{9500}{2499}\frac{k}{L^2}x^2$), at L/2 with a length of 2m, the foundation stiffness reaches the given K value:*

Table 6. Case 6 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	-5.908.10^{-4} (m)	-5.037.10^{-4} (m)	-5.9.10^{-4} (m)
Viscoelastic	-5.884.10^{-4} (m)	-5.019.10^{-4} (m)	-5.876.10^{-4} (m)
Difference	0.41%	0.36%	0.41%

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 13 and Fig. 14.

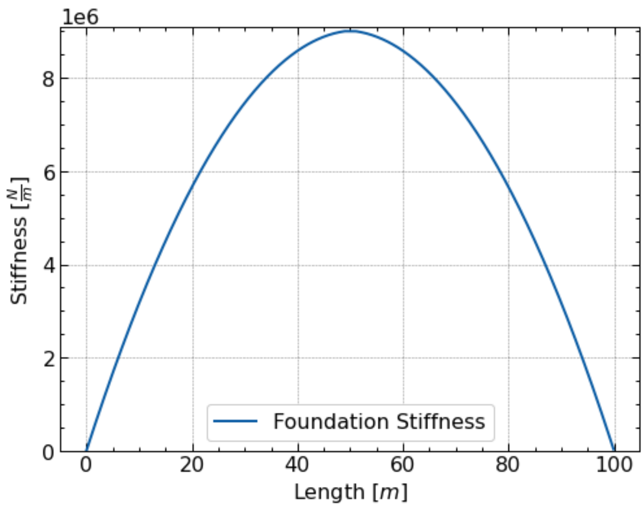

Fig. 13. Case 6 – Stiffness

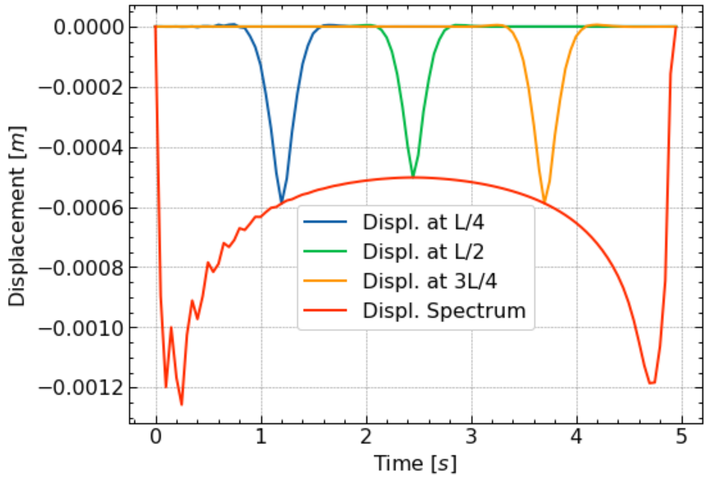

Fig. 14. Case 6 – Displacement

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 6.

Remark: this is a relatively common case in the practice of plate-on-foundation structure. As described in Case 1, at time t=0 the load is in contact with the plate element at high velocity, giving the plate element a non-zero acceleration along with the mass of the plate-foundation element causing the force inertia and producing a relatively large displacement at the initial instant. In addition, the foundation at the transition position between the joint and the plate is depressed (K=0) which can easily cause damage to the plate structure. In practice, we can observe the transitional positions between the abutment and the road, which are often damaged leading to the vehicle moving not smoothly. Besides, from the displacement spectrum and displacement plane, we can see that the location due to the load acting when entering the plate will be larger due to the dynamic coefficient when compared to the location due to the load acting when leaving the plate.

Case 7*, the foundation has a stiffness $K = 0$ (depressed at the joint) and follows the law of sine function ($k_f(x) = \frac{-k}{2} sin\left(\frac{-8\pi}{L}x + \frac{\pi}{2}\right) + \frac{k}{2}$):*

The foundation stiffness and the displacement of the dynamic foundation are shown in Fig. 15 and Fig. 16.

The comparison of displacement between the dynamic foundation and the viscoelastic foundation is shown in Table 7.

Table 7. Case 7 – Displacement

Foundation model	Displacement at L/4	Displacement at L/2	Displacement at 3L/4
Dynamic	$-19.723.10^{-4}$ (m)	$-19.706.10^{-4}$ (m)	$-19.706.10^{-4}$ (m)
Viscoelastic	$-19.056.10^{-4}$ (m)	$-19.062.10^{-4}$ (m)	$-19.063.10^{-4}$ (m)
Difference	3.38%	3.27%	3.26%

Remark: this is the general case in practice. This case can be considered as including all the above cases.

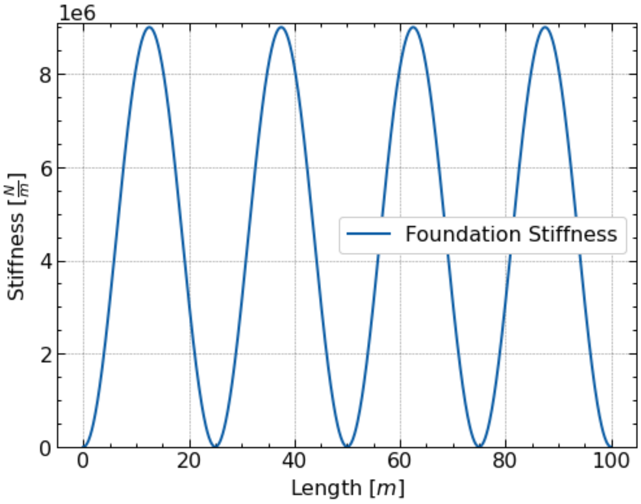

Fig. 15. Case 7 – Stiffness

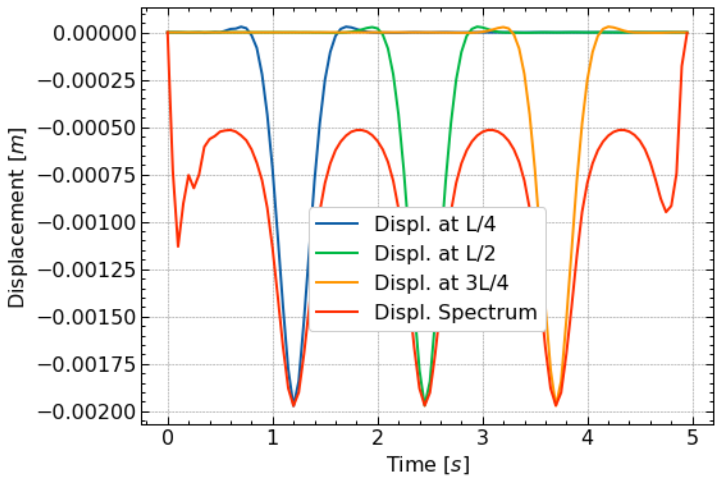

Fig. 16. Case 7 – Displacement

Discussion

From the above cases, the dynamic foundation model in the case of a weakening foundation will give a significantly larger displacement compared to the massless foundation model (in this case, the viscoelastic foundation is considered).

In addition, if a correlative comparison is made between the foundation problem with the constant stiffness coefficient, that is the foundation model commonly found in design calculations, and the analyzed cases, the results will be correlated as shown in Table 8.

Table 8. Correlation between constant foundation stiffness and other cases

	L/4 (10^{-4}m)	L/2 (10^{-4}m)	3L/4 (10^{-4}m)
Case 1	-5.022	-5.033	-5.033
Case 2	70%	70%	70%
Case 3	31%	239%	31%
Case 4	45%	630%	45%
Case 5	100%	1911%	101%
Case 6	18%	0%	17%
Case 7	293%	292%	292%

With the analysis of variable foundation stiffness, that is the foundation changes and leads to weakening during the operation, all show larger displacements than the foundation with constant stiffness. Therefore, in the studies that take the foundation with constant stiffness and masslessness, this issue should also be considered.

4 Conclusion

The foundation needs to be uniform so that the plate-on-foundation structure can work at its best, we can use locally available and suitable materials, no need to search for too good materials to avoid waste. The need to apply design methods as well as construction techniques to avoid the depression of the foundation stiffness, and the settlement of the foundation due to seepage, flooding, and water flow. So, these damages could be difficult to repair.

The position of transition from the joint to the foundation is a "weak" position in practice, this is the part that should be reinforced compared to the design calculation.

The locations where the foundation is depressed, or the stiffness is reduced, should be rectified early with appropriate solutions to minimize the damage to the structure and ensure traffic safety and the comfort of vehicle movement.

The dynamic foundation model gives analytical results closer to reality than the foundation models that do not consider mass.

This article has analyzed plate behavior on a dynamic foundation with variable stiffness, but these analyzes are still very basic, many issues need to be further investigated, such as variable load velocity, shock force when the load is not continuously in contact with the surface of the plate, the multi-layer foundation, the braking force, multi-parameter foundation.

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