Free Vibration Analysis of Prestressed Timoshenko Beams Using the Modal Analysis Method

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Abstract. Natural frequencies and mode shapes are important features in the static analysis of beam structures. They are used in analysis, design as well as verification of beam structures. They also play an important role in influencing the dynamic response of beams. For example, just changing the natural frequency by a certain amount will cause the forced vibration of the structure from the near-resonance region to out of the near-resonant region. Therefore, in this study, the modal analysis method is used to establish the characteristic equations to determine the natural frequencies and the vibration modal functions for prestressed beams with different boundary conditions. By numerical method, this work analyzed and compared these natural frequencies with the natural frequencies of ordinary Timoshenko beams and Euler - Bernoulli beams.

Keywords: Natural Frequency, Prestressed Beam, Timoshenko Beam, Modal Function.

1 Introduction

For a long time, Timoshenko (**TM**) beam theory is widely used [1-5] for static and dynamic analysis of elastic structures, such as beams, concrete bridges, railway bridges, etc. Euler – Bernoulli beam (**EB**) theory has ignored shear deformation and rotational inertia, then TM beam theory includes both factors. Therefore, it is not possible to establish an analytic relationship between shear force and moment in the TM beam. TM beams are also known as thick beam theory or second-order beam theory.

If the transverse bending vibration of the EB beam is described by a partial differential equation of deflection, then the vibrational equations of the TM beam are a system of second-order partial differential equations in which the amplitude of vibration and the angle of rotation due to pure bending are the values to be determined [6-8]:

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}-\frac{\partial Q(x,t)}{\partial x}=q(x,t) \quad (1)$$

$$\rho I(x)\frac{\partial^2 \phi(x,t)}{\partial t^2}+\frac{\partial M(x,t)}{\partial x}-Q(x,t)=\tau(x,t) \quad (2)$$

To analyze the natural frequencies and the mode shapes, it is common to use the free vibration equations (the right hand side is zero). To analyze forced vibration, the right-hand side equations are often used. Methods of studying these problems have many types such as Ritz method, finite element method (**FEM**), finite difference method (**FDM**), modal analysis method (**MAM**), etc.

In the problems of structural verification and evaluation, the natural frequencies are the characteristics that need to be determined. Natural frequency and natural form are two characteristics of beams that are also used to analyze forced vibrations of beams [6-10] using MAM. Moreover, in the dynamics problem, when changing frequencies will affect its oscillation properties, it is possible that the oscillation is moving from the far-resonant region to the near-resonant region or vice versa. Therefore, it is necessary to define them.

For ordinary beams, these characteristics have been studied quite fully. With prestressed EB beams has been interested in many studies [11-17]. In [13], the effect of longitudinal force on natural frequency and mode shapes of prestressed EB beam was investigated according to a beam model on two supports. The effect of "softening of beams by pre-compression" representing the reduction of bending natural frequencies of beams due to pre-compression was mentioned in [14]. Research [15] used FEM combined with experiment to evaluate the effect of prestress on natural frequency of beams with free boundary. The study [16] determined the natural frequency and eigenform of the EB beam with cracks and prestressing, then in [17] they investigated the dynamics response of that type of beam under the action of a moving body.

For prestressed TM beams (Prestressed Timoshenko - **PTM**), studies are still limited, In Ref. [18] the authors use the reverberation-ray matrix method to study the vibrations of prestressed Rayleigh-Timoshenko beams subjected to arbitrary forces. Ref. [19] To study the free vibration of a PTM beam placed on a Winkler elastic foundation using FEM. The natural frequency of the beam is not determined.

Prestressed TM beams due to tension or compression are very common in practice, especially in concrete structures. Changing the natural frequency law in the presence of prestressing factors is necessary to be considered. In this study, this problem will be considered and compared with the theories of normal TM beams or EB beams. To determine the natural frequencies and mode functions, the paper will set up for the general case. Then, by substituting the boundary conditions into the general form, we obtain the formula for each beam corresponding to its boundary conditions. The characteristic equations that determine the natural frequencies of the beams are usually nonlinear algebraic equations, which can be solved by the Newton-Raphson method.

2 General Equations

2.1 Vibration equations of the prestressed Timoshenko beam

Consider the PTM beam model as shown in Fig.1. The geometric axis of the beam is straight without deformation. Neglect torsional and axial vibrations. Beams only perform bending vibrations and under the action of distributed force. Thus, transverse displacement, cross-sectional rotation, bending moment and shear force are functions of the *x*-coordinate and time *t*:

$$w(x,t),\varphi(x,t),M(x,t),Q(x,t) \tag{3}$$

For the TM beam, the rotation angle φ of the cross-section is equal to the sum of the rotation angle ϕ caused by the bending strain and the rotation angle γ caused by the shear strain. For EB beams, the angle component γ is zero due to ignoring the effect of shear strain.

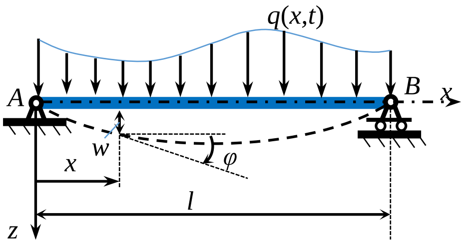


Fig. 1. Beam subjected to distributed force $q(x,t)$

$$\tan\varphi = \frac{\partial w}{\partial x} \approx \varphi = \phi + \gamma$$

The transverse vibration equations of the beam [6-9] has the form:

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial Q(x,t)}{\partial x} = q(x,t) \tag{4}$$

$$\rho I(x)\frac{\partial^2 \phi(x,t)}{\partial t^2} + \frac{\partial M(x,t)}{\partial x} - Q(x,t) = \tau(x,t) \tag{5}$$

where $A(x)$ is the cross -sectional area, ρ is the mass density.

Bending moment and shear force determined by the formula

$$M = \int_A z\sigma_{xx}dA,\ Q = \kappa GA(x)\gamma = \kappa GA(x)\left(\frac{\partial w}{\partial x} - \phi\right) \tag{6}$$

where G is the shear modulus, κ is the correction shear factor.

The case of TM beams. The axial stress is determined by Hook's law in the form

$$\sigma_{xx} = E\varepsilon_{xx}(x,z,t),\ \varepsilon_{xx}(x,z,t) = -z\frac{\partial\phi}{\partial x} \tag{7}$$

where E is Young's modulus, ε_{xx} is the normal strain of the beam due to bending displacement.

Substituting expressions (7), (6) into equations (4), (5) we get:

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2} - \kappa GA\left(\frac{\partial^2 w(x,t)}{\partial^2 x} - \frac{\partial\phi(x,t)}{\partial x}\right) = q(x,t) \tag{8}$$

$$\rho I\frac{\partial^2 \phi(x,t)}{\partial t^2} = \kappa GA\left(\frac{\partial w(x,t)}{\partial x} - \phi\right) + EI\frac{\partial^2\phi(x,t)}{\partial x^2} \tag{9}$$

Thus, we obtain two of partial differential equations of ordinary Timoshenko beams with two unknowns $w(x,t)$ and $\phi(x,t)$

The case of PTM beams. Normal strain $\varepsilon_{xx}(x,z,t)$ in the *x*-direction:

$$\varepsilon_{xx}(x,z,t) = \varepsilon_0(x) + \varepsilon^*_{xx}(x,z,t) \tag{10}$$

where $\varepsilon_0(x)$ is the initial normal strain caused by prestressing. Considering the position of the cross-section with coordinates x, the proportional long strain is:

$$\varepsilon_0(x,z,0)=\varepsilon_0(x)\,, \tag{11}$$

$\varepsilon_{xx}^{*}(x,z,t)$ is the normal strain added when the beam is deformed. In the elastic domain, these strains are linear, the total normal strain is equal to the initial strain due to prestressing and the strain due to bending.

Considering a flexural beam element as shown in Fig.2, the relative long strain of a layer z any of the beams depends on the coordinates z, when $z=z_0$ then $\varepsilon(z_0)=0$.

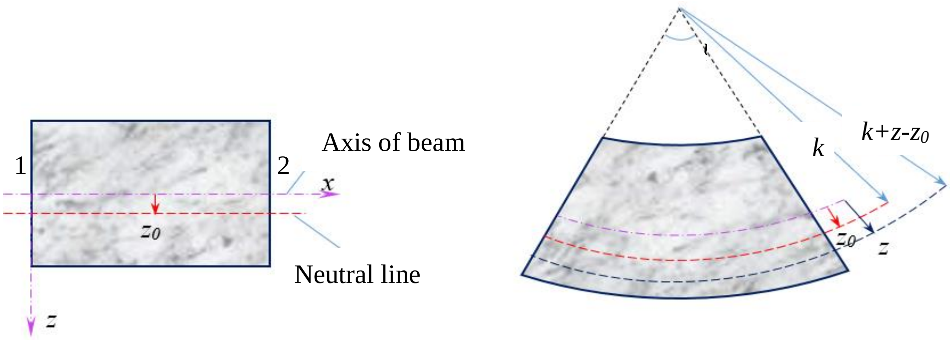


Fig. 2. Normal strain of a beam segment

For homogeneous and symmetrical beams, the neutral line coincides with the symmetry axis. However, in the general case the neutral line does not coincide with the axis of symmetry of the beam. The symbol k is the radius of curvature of the neutral line z_0. Since the neutral line is not long deformed, the initial distance between the two cross-sections in terms of the neutral line is:

$$L_0=k\psi \tag{12}$$

The distance between the point at section 1 and the point at section 2 of the material layer with z-coordinate after deformation will be:

$$L=(k+z-z_0)\psi \tag{13}$$

From the formula (12) and (13), consider the case that the neutral axis coincides with the symmetry axis, ie $z_0=0$, we deduce the relative strain of the material layer with z-coordinate is

$$\varepsilon_{xx}^{*}(z)=\frac{L-L_0}{L_0}=\frac{z}{k} \tag{14}$$

where the curvature k of the beam is determined by the mathematical formula:

$$\frac{1}{k}=-\frac{\partial^2 w}{\partial x^2}\Big/\left(1+\left(\frac{\partial w}{\partial x}\right)^2\right)^{\frac{3}{2}} \tag{15}$$

Substituting (15) into (14), ignoring the higher-order infinity, we get:

$$\varepsilon_{xx}(x,z,t)\approx\varepsilon_0(x)-\frac{\partial^2 w}{\partial x^2}z \tag{16}$$

Substituting (16) into (6) we get:

$$M = \varepsilon_0(x)E\int_A z dA - E\frac{\partial^2 w}{\partial x^2}\int_A z^2 dA \tag{17}$$

We have the formula for determining the center of gravity of the *dx*-element and the inertia moment of the cross-section:

$$Az_C = \int_A z dA, \quad I(x) = \int_A z^2 dA \tag{18}$$

where z_C is the coordinates of the center of gravity in the *z*-axis, when the beam is deformed, then $z_C = w$.

Substituting (18) into (17), we get:

$$M(x,t) = E\varepsilon_0(x)A(x)w - EI(x)\frac{\partial^2 \mathrm{w}}{\partial x^2} \tag{19}$$

Substituting (19) and (6) into equations (4), (5) one obtains:

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[\kappa GA(x)\left(\frac{\partial w}{\partial x} - \phi\right)\right] = q(x,t) \tag{20}$$

$$\rho I(x)\frac{\partial^2 \phi(x,t)}{\partial t^2} + \frac{\partial}{\partial x}\left(E\varepsilon_0(x)A(x)w - EI(x)\frac{\partial^2 \mathrm{w}}{\partial x^2}\right) - \kappa GA(x)\left(\frac{\partial w}{\partial x} - \phi\right) = 0 \tag{21}$$

Assume that the beam is homogeneous, the cross-section is constant, the neutral axis coincides with the symmetry axis, then A, I, ε_0 these are constants. The motion equations have the form:

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2} - \kappa GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) = q(x,t) \tag{22}$$

$$\rho I\frac{\partial^2 \phi(x,t)}{\partial t^2} + \varepsilon_0 EA\frac{\partial w}{\partial x} - EI\frac{\partial^3 w}{\partial x^3} - \kappa GA\left(\frac{\partial w}{\partial x} - \phi\right) = 0 \tag{23}$$

2.2 Eigenvalues and Mode Functions

Free vibration equations of the PTM beam has the form:

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2} - \kappa GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) = 0 \tag{24}$$

$$\rho I\frac{\partial^2 \phi(x,t)}{\partial t^2} + \varepsilon_0 EA\frac{\partial w}{\partial x} - EI\frac{\partial^3 w}{\partial x^3} - \kappa GA\left(\frac{\partial w}{\partial x} - \phi\right) = 0 \tag{25}$$

For normal TM beams, their solutions have the variable dissociation [6,7], according to the expression:

$$w(x,t) = \mathrm{W}(x).T(t) \tag{26}$$

$$\phi(x,t) = \varPhi(x).T(t) \tag{27}$$

For PTM beams, this needs to be proven*:*

Lemma proving the ability to dissociation solutions. Derivative Eq. (25) according to the variable *x*, then substituting equation (21) into it we get:

$$\rho I\frac{\partial^3\phi(x,t)}{\partial x\partial t^2}+\varepsilon_0 EA\frac{\partial w^2(x,t)}{\partial x^2}-EI\frac{\partial\phi^3(x,t)}{\partial x^3}-\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{28}$$

From Eq. (24) we can also derive:

$$\frac{\partial\phi}{\partial x}=\frac{\partial^2 w}{\partial x^2}-\frac{\rho A}{\kappa GA}\frac{\partial^2 w(x,t)}{\partial t^2} \tag{29}$$

From Eq. (29) we calculate the derivatives:

$$\frac{\partial^3\phi}{\partial x^3}=\frac{\partial^4 w}{\partial x^4}-\frac{\rho A}{\kappa GA}\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2},\quad \frac{\partial^3\phi}{\partial x\partial t^2}=\frac{\partial^4 w}{\partial x^2\partial t^2}-\frac{\rho A}{\kappa GA}\frac{\partial^4 w(x,t)}{\partial t^4} \tag{30}$$

Substituting (30) into (28) we have:

$$\rho I\left(\frac{\partial^4 w}{\partial x^2\partial t^2}-\frac{\rho A}{\kappa GA}\frac{\partial^4 w(x,t)}{\partial t^4}\right)+\varepsilon_0 EA\frac{\partial^2 w(x,t)}{\partial x^2}-EI\left(\frac{\partial^4 w}{\partial x^4}-\frac{\rho A}{\kappa GA}\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2}\right)-\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0$$

Simplifying we get a partial derivative of fourth order of $w(x,t)$:

$$-EI\frac{\partial^4 w}{\partial x^4}+\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2}-\frac{\rho^2 I}{\kappa G}\frac{\partial^4 w(x,t)}{\partial t^4}+\varepsilon_0 EA\frac{\partial w^2(x,t)}{\partial x^2}-\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{31}$$

Continuing the first-order derivative of Eq.(21) with respect to the variable *x* we get

$$\rho A\frac{\partial^3 w}{\partial x\partial t^2}-\kappa GA\left(\frac{\partial^3 w}{\partial x^3}-\frac{\partial^2\phi}{\partial x^2}\right)=0 \tag{32}$$

From (23) we derive the components and their derivatives and then substitute (32), the same transformation we get the equation:

$$-EI\frac{\partial^4\phi}{\partial x^4}+\rho I\left(\frac{E}{\kappa G}+1\right)\frac{\partial^4\phi}{\partial t^2\partial x^2}-\frac{\rho^2 I}{\kappa G}\frac{\partial^4\phi}{\partial t^4}+\varepsilon_0 EA\frac{\partial^2\phi}{\partial x^2}-\rho A\frac{\partial^2\phi}{\partial t^2}=0 \tag{33}$$

Thus, equations (31) and (33) are dissociation equations written for each variable. Moreover, they have the same form. Thus the solutions (26) and (27) are still valid for RTM beams.

Eigenvalues and Mode Functions. Derivative (26), (27) then substituting into the equation (24), (25) transform we get:

$$\frac{\kappa G}{\rho}\frac{W''(x)}{W(x)}-\frac{\kappa G}{\rho}\frac{\Phi'(x)}{W(x)}=\frac{\ddot{T}(t)}{T(t)} \tag{34}$$

$$-\frac{\varepsilon_0 EAW'(x)}{\rho I\Phi(x)}+\frac{EI\Phi''(x)}{\rho I\Phi(x)}+\frac{\kappa GAW'(x)}{\rho I\Phi(x)}-\frac{\kappa GA}{\rho I}=\frac{\ddot{T}(t)}{T(t)} \tag{35}$$

Since the right sides of equations depend on *t* and the left sides of the equations depend on *x*, the two sides must be constants.

Put

$$\frac{\ddot{T}(t)}{T(t)}=-\omega^2 \tag{36}$$

Substituting (36) into Eq.(34) and Eq.(35), transform we get:

$$\kappa G W''(x) - \kappa G \Phi'(x) + \omega^2 \rho W(x) = 0 \tag{37}$$

$$EI\Phi''(x) + \left(\kappa GA - \varepsilon_0 EA\right) W'(x) + \left(\omega^2 \rho I - \kappa GA\right) \Phi(x) = 0 \tag{38}$$

From Eq. (37) we have:

$$\Phi'(x) = W''(x) + \frac{\omega^2 \rho}{\kappa G} W(x) \tag{39}$$

Deriving Eq.(38) with respect to x and then substituting (39) into it, we get the equation:

$$\mathrm{W}^{(4)} + \left(\left(\frac{1}{\kappa G} + \frac{1}{E} \right) \rho \omega^2 - \frac{\varepsilon_0 A}{I} \right) \mathrm{W}'' + \omega^2 \rho \left(\frac{\omega^2 \rho}{E\kappa G} - \frac{A}{EI} \right) \mathrm{W} = 0 \tag{40}$$

The characteristic Eq.(40) is:

$$\lambda^4 + b\lambda^2 + c = 0 \tag{41}$$

Where

$$b = \left(\frac{1}{\kappa G} + \frac{1}{E} \right) \rho \omega^2 - \frac{\varepsilon_0 A}{I}, \; c = \rho \omega^2 \left(\frac{\omega^2 \rho}{E\kappa G} - \frac{A}{EI} \right) \tag{42}$$

Replacing $k = \lambda^2$, we have the equation:

$$k^2 + bk + c = 0 \tag{43}$$

Case 1. c = 0:

$$\Rightarrow \frac{\omega^2 \rho}{E\kappa G} - \frac{A}{EI} = 0 \;\; \Rightarrow \;\; \omega = \omega_c = \sqrt{\frac{\kappa GA}{\rho I}} \tag{44}$$

That ω_c is called *cut-off* frequency. This is a mathematical value, practically no natural frequency of a beam will equal this value.

Case 2. $c \neq 0$. Its delta determinant is positive definite: $\Delta = \mathrm{b}^2 - 4c > 0$

Indeed

$$\Delta = \left(\left(\frac{1}{\kappa G} + \frac{1}{E} \right) \rho \omega^2 - \frac{\varepsilon_0 A}{I} \right)^2 - 4\omega^2 \rho \left(\frac{\omega^2 \rho}{E\kappa G} - \frac{A}{EI} \right)$$

$$\Leftrightarrow \Delta = \left(\frac{\varepsilon_0 A}{I} - \left(\frac{1}{\kappa G} - \frac{1}{E} \right) \rho \omega^2 \right)^2 + \frac{4A}{EI} \left(1 - \varepsilon_0\right) \rho \omega^2 > 0$$

Due to $\left|\varepsilon_0\right| \ll 1$, then $\Delta > 0$. The solutions of Eq. (43) are:

$$k_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, k_2 = \frac{-b - \sqrt{b^2 - 4c}}{2} \tag{45}$$

We consider two frequency regions:

Frequency region $\omega < \omega_c$. From (44) and (42) deduce $c < 0$. From $k = \lambda^2$ we have:

With $k_1 = \dfrac{-b + \sqrt{b^2 - 4c}}{2} > 0 \;\Rightarrow\; \lambda_1 = \beta, \; \lambda_2 = -\beta \; (\beta > 0)$ (46)

With $k_2 = \dfrac{-b - \sqrt{b^2 - 4c}}{2} < 0 \Rightarrow \lambda_1 = i\xi, \; \lambda_2 = -i\xi$ (47)

where

$$\beta = \frac{\sqrt{-b+\sqrt{b^2-4c}}}{\sqrt{2}}; \quad \xi = \frac{\sqrt{b+\sqrt{b^2-4c}}}{\sqrt{2}} \tag{48}$$

Frequency region $\omega > \omega_c$. From (44) and (42) we deduce $c < 0$. We have:

With $k_1 = \dfrac{-b+\sqrt{b^2-4c}}{2} < 0 \Rightarrow \lambda_1 = i\eta,\ \lambda_2 = -i\eta$

With $k_2 = \dfrac{-b-\sqrt{b^2-4c}}{2} < 0 \Rightarrow \lambda_3 = i\xi,\ \lambda_4 = -i\xi$

where

$$\eta = \frac{\sqrt{b-\sqrt{b^2-4c}}}{\sqrt{2}} \tag{49}$$

The equations (37) and (38) having solutions are determined according to the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as follows:

Case $\omega < \omega_c$. Solutions of the form

$$W(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x} = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\xi x} + C_4 e^{-i\xi x} \tag{50}$$

$$\Phi(x) = C_1' e^{\lambda_1 x} + C_2' e^{\lambda_2 x} + C_3' e^{\lambda_3 x} + C_4' e^{\lambda_4 x} = C_1' e^{\beta x} + C_2' e^{-\beta x} + C_3' e^{i\xi x} + C_4' e^{-i\xi x} \tag{51}$$

Using Euler's formulas

$$e^{i\xi x} = \cos\xi x + i\sin\xi x,\ e^{-i\xi x} = \cos\xi x - i\sin\xi x,\ \sinh\beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}, \cosh\beta x = \frac{e^{\beta x} + e^{-\beta x}}{2}$$

We get the trigonometric form:

$$W(x) = a_1 \sinh\beta x + a_2 \cosh\beta x + a_3 \sin\xi x + a_4 \cos\xi x \tag{52}$$

$$\Phi(x) = d_1 \sinh\beta x + d_2 \cosh\beta x + d_3 \sin\xi x + d_4 \cos\xi x \tag{53}$$

where a_i and d_i (i = 1, 2, 3, 4) are constants depending on the boundary conditions of the beams. The functions $W(x)$ and $\Phi(x)$ are not independent, so the constants a_i and d_i are also not independent.

Substituting (52), (53) into (37), identifying the trigonometric functions on both sides, we get:

$$\left(\kappa G a_1 \beta^2 - \kappa G d_2 \beta + \omega^2 \rho\ a_1\right) = 0 \Rightarrow d_2 = a_1\left(\beta + \frac{\omega^2 \rho}{\kappa G \beta}\right) = a_1 h_\beta \tag{54}$$

$$\left(\kappa G a_2 \beta^2 - \kappa G d_1 \beta + \omega^2 \rho\ a_2\right) = 0 \Rightarrow d_1 = a_2\left(\beta + \frac{\omega^2 \rho}{\kappa G \beta}\right) = a_2 h_\beta \tag{55}$$

$$\left(-a_3 \xi^2 \kappa G + \kappa G d_4 \xi + \omega^2 \rho\ a_3\right) = 0 \Rightarrow d_4 = a_3\left(\xi - \frac{\omega^2 \rho}{\kappa G \xi}\right) = a_3 f_\xi \tag{56}$$

$$\left(-\kappa G a_4 \xi^2 - \kappa G d_3 \xi + \omega^2 \rho\ a_4\right) = 0 \Rightarrow d_3 = -a_4\left(\xi - \frac{\omega^2 \rho}{\kappa G \xi}\right) = -a_4 f_\xi \tag{57}$$

where $$h_\beta = \left(\beta + \frac{\omega^2\rho}{\kappa G\beta}\right),\ f_\xi = \left(\xi - \frac{\omega^2\rho}{\kappa G\xi}\right) \tag{58}$$

Substituting equations from (54) to (57) into (53) we get the mode functions:

$$W(x) = a_1 \sinh\beta x + a_2 \cosh\beta x + a_3 \sin\xi x + a_4 \cos\xi x \tag{59}$$

$$\Phi(x) = a_2 h_\beta \sinh\beta x + a_1 h_\beta \cosh\beta x - a_4 f_\xi \sin\xi x + a_3 f_\xi \cos\xi x \tag{60}$$

Case $\omega > \omega_c$. Solutions of the form

$$\mathrm{W}(x) = C_1 e^{i\eta x} + C_2 e^{-i\eta x} + C_3 e^{i\xi x} + C_4 e^{-i\xi x}$$

$$\Phi(x) = C_1' e^{i\eta x} + C_2' e^{-i\eta x} + C_3' e^{i\xi x} + C_4' e^{-i\xi x}$$

Using Euler's formulas to trigonometric form and similar transformations, we get mode functions:

$$W(x) = b_1 \sin\eta x + b_2 \cos\eta x + b_3 \sin\xi x + b_4 \cos\xi x \tag{61}$$

$$\Phi(x) = -b_2 f_\eta \sin\eta x + b_1 f_\eta \cos\eta x - b_4 f_\xi \sin\xi x + b_3 f_\xi \cos\xi x \tag{62}$$

where

$$f_\eta = \left(\eta - \frac{\omega^2\rho}{\kappa G\eta}\right) \tag{63}$$

Thus, the mode functions of the PTM beam is determined by formulas (59) and (60) in case $\omega < \omega_c$, or by formulas (61) and (62) in case $\omega < \omega_c$.

3 Calculation of Natural Frequencies and Mode Shapes

3.1 Simply Supported PTM Beam

The boundary conditions of simply supported (**SS**) PTM beam are both transverse displacement and bending moment equal zeros at $x = 0$ and $x = l$:

$$w(0,t) = 0\,;\ M(0,t) = \varepsilon_0 EAw(0,t) - EI\frac{\partial\phi(0,t)}{\partial x} = 0 \tag{64}$$

$$w(l,t) = 0\,;\ M(l,t) = \varepsilon_0 EAw(l,t) - EI\frac{\partial\phi(l,t)}{\partial x} = 0 \tag{65}$$

Substituting (26) and (27) into (64), (65) we obtain:

$$W(0) = 0,\ \Phi'(0) = 0,\ W(l) = 0,\ \Phi'(l) = 0 \tag{66}$$

Case $\omega < \omega_c$

Natural frequencies. Substituting the Eqs. (59), (60) into (66) we get:

$$W(0) = a_2 + a_4 = 0 \tag{67}$$

$$\Phi'(0) = \beta a_2 h_\beta - \xi a_4 f_\xi = 0 \tag{68}$$

$$W(l) = a_1 \sinh\beta l + a_2 \cosh\beta l + a_3 \sin\xi l + a_4 \cos\xi l = 0 \tag{69}$$

$$\Phi'(l) = \beta a_2 h_\beta \cosh\beta l + \beta a_1 h_\beta \sinh\beta l - \xi a_4 f_\xi \cos\xi l - \xi a_3 f_\xi \sin\xi l = 0 \tag{70}$$

From (67), (68) we have:

$$a_2 = a_4 = 0 \tag{71}$$

Substituting (71) into (69) and (70) we get

$$a_1 \sinh \beta l + a_3 \sin \xi l = 0 \tag{72}$$

$$\beta a_1 h_\beta \sinh \beta l - \xi a_3 f_\xi \sin \xi l = 0 \tag{73}$$

For the solutions a_1 and a_3 are non-zero, then there must have:

$$\begin{vmatrix} \sinh \beta l & \sin \xi l \\ \beta h_\beta \sinh \beta l & -\xi f_\xi \sin \xi l \end{vmatrix} = 0 \Rightarrow \left(\xi f_\xi + \beta h_\beta \right) \sin \xi l \sinh \beta l = 0 \tag{74}$$

From the formula (48) and (58) we obtain

$$\xi f_\xi + \beta h_\beta = 2\sqrt{\Delta} > 0\ . \tag{75}$$

From (46), because $\beta > 0$ so $\sinh \beta l > 0$. Therefore, formula (74) is obtained

$$\sin \xi l = 0 \Rightarrow \xi = \frac{j\pi}{l} \ \left(j = 1, 2, ..., N_L \right) \tag{76}$$

Substituting (48) into Eq.(76) and solving it we get the natural frequencies $\omega_j < \omega_c$

Mode shapes. Multiply both sides of the Eq. (72) by ξf_ξ and then add to (73) we get:

$$a_1 \left(\xi f_\xi + \beta h_\beta \right) \sinh \beta l = 0$$

Because $\left(\xi f_\xi + \beta h_\beta \right) > 0$, $\sinh \beta l > 0$ so inferred $a_1 = 0$. Therefore, to have a solution $a_i \neq 0$ then $a_3 \neq 0$. Substituting $a_1 = a_2 = a_4 = 0$, $a_3 \neq 0$ into (59), (60) we get the mode shapes:

$$W(x) = a_3 \sin \frac{j\pi}{l} x\ ,\ \Phi(x) = a_3 f_\xi \cos \frac{j\pi}{l} x \tag{77}$$

Case $\omega > \omega_c$

Natural frequency. Substituting the Eq. (61) and Eq. (62) into (66), the same proof we get:

$$\sin \eta l = 0 \Rightarrow \eta = \frac{n\pi}{l} \ \left(n = 1, 2, ..., \right) \tag{78}$$

Substituting (49) into Eq.(78) and solving it we get the natural frequencies $\omega_n > \omega_c$

Mode shapes. From formulas (61) and (62) similarly, we have:

$$W(x) = b_1 \sin \frac{n\pi}{l} x\ ,\quad \Phi(x) = b_1 f_\eta \cos \frac{n\pi}{l} x \tag{79}$$

Numerical results. Numerical calculation is performed with the set of parameters as [6]. The properties of the beam are: length l = 1 m, cross-section with width b = 0.02 m and height h = 0.08 m, E = 2.1x10^{11} N/m^2, G = 8.1x10^{10} N/m^2, ρ = 7860 kg/m^3, κ = 0.5. We can determine the cut-off frequency according to the formula (44) is $\omega_c = 98291.706$ rad/s (15643.62 Hz) and determines the number of natural frequencies less than the cut-off frequency is $N_B = 73$.

To compare the numerical results with EB beam, we have the formula to determine the natural frequencies of EB beams [20]:

$$\omega_k = \left(\frac{k\pi}{l}\right)^2 \sqrt{\frac{EI}{\rho A}}, \text{ with } k = 1, 2, 3\ldots$$

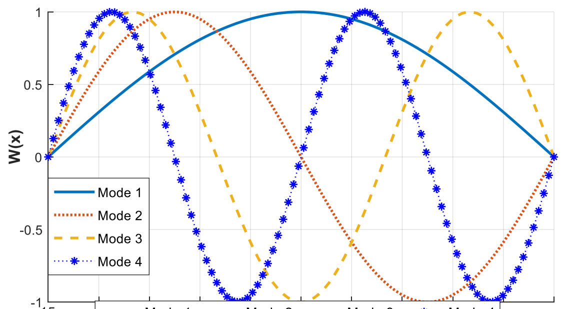
The mode shapes of displacement and rotation of SS PTM beam as shown in Fig.3 and Fig.4. Calculation results of natural frequencies presented in Table 1.

Table 1. Natural frequencies of SS PTM Beam

	EB [20] (rad/s)	**TM** (rad/s) $\varepsilon_0 = 0$ **(a)**	**[6]**	**PTM** (rad/s) $\varepsilon_0 = -10^{-4}$	$\varepsilon_0 = -5.10^{-4}$ **(b)**	$\varepsilon_0 = 10^{-4}$	**(b-a)/a** (%)
1	1178.141	1159.497	1159.4	1148.428	1103.046	1170.460	4.87
2	4712.566	4436.759	4436.8	4426.191	4383.668	4447.301	1.20
3	10603.274	9357.616	9357.6	9347.665	9307.755	9367.558	0.53
4	18850.265	15409.984	15410.0	15400.680	15363.412	15419.283	0.30
5	29453.539	22182.504	22183.0	22173.824	22139.071	22191.182	0.20
6	42413.096	29389.344	-	29381.241	29348.809	29397.446	0.14
7	57728.937	36845.603	-	36838.026	36807.706	36853.178	0.10
8	75401.061	44435.880	-	44428.781	44400.375	44442.979	0.08
9	95429.468	52089.639	-	52082.972	52056.300	52096.305	0.06
10	117814.15	59764.516	-	59758.242	59733.142	59770.789	0.05
15	265081.85	97885.610	-	97880.832	97861.716	97890.389	0.03

From Table 2 it can be seen that: When the beam is subjected to pre-compression ($\varepsilon_0 < 0$), the natural frequencies of the PTM beam are reduced compared to that of the normal TM beam. The more compressed the beam, the lower the frequencies. This is called "softening" in prestressed beams. When the beam is in pre-tensioned ($\varepsilon_0 > 0$), its natural frequency increases compared to the normal beam.

The natural frequencies of the PTM beams compared with the natural frequencies of the TM beams differ the most at



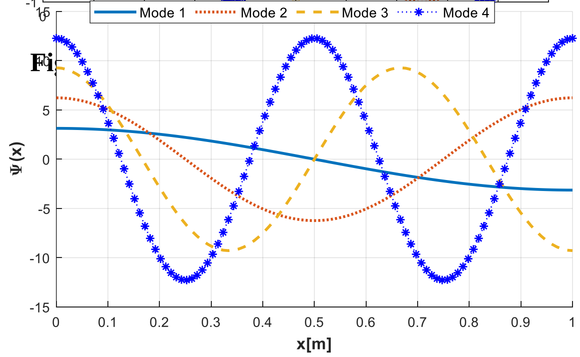


Fig. 4. Mode shapes of cross-sectional rotation

the first frequency (about 4.9% corresponding to $\varepsilon_0 = -5\times10^{-4}$). It decreases rapidly as the frequency order increases. PTM beams when subjected to pre-compression ($\varepsilon_0 < 0$) deviate from EB beams more than when subjected to pre-tension ($\varepsilon_0 > 0$).

3.2 Clamped – Free PTM Beam

Boundary conditions of clamped – free (**CF**) beams

At $x = 0$, displacement and rotation are both 0:

$$\begin{aligned} &w(0,t) = 0 \Rightarrow W(0) = 0; \\ &\phi(0,t) = 0 \Rightarrow \Phi(0) = 0 \end{aligned} \tag{80}$$

At $x = l$, torque is 0, shear force is 0, similarly we have:

$$\begin{aligned} &\frac{\partial\phi(l,t)}{\partial x} = 0 \Rightarrow \Phi'(l) = 0 \\ &k^* GA(l)[w'(l,t) - \phi(l,t)] = 0 \Rightarrow \mathrm{W}'(l) - \Phi(l) = 0; \end{aligned} \tag{81}$$

Case $\omega < \omega_c$

Natural frequency. Substituting the Eqs. (59), (60) into (80), (81) we get:

$$a_2 + a_4 = 0 \tag{82}$$

$$a_1 h_\beta + a_3 f_\xi = 0 \tag{83}$$

$$\beta a_2 h_\beta \cosh\beta l + \beta a_1 h_\beta \sinh\beta l - \xi a_4 f_\xi \cos\xi l - \xi a_3 f_\xi \sin\xi l = 0 \tag{84}$$

$$a_1\left(\beta - h_\beta\right)\cosh\beta l + a_2\left(\beta - h_\beta\right)\sinh\beta l + a_3\left(\xi - f_\xi\right)\cos\xi l - a_4\left(\xi - f_\xi\right)\sin\xi l = 0 \tag{85}$$

From (82), (83) we get:

$$a_4 = -a_2,\ a_3 = -a_1 h_\beta / f_\xi \tag{86}$$

Substituting (86) into (84) and (85) we get:

$$\beta a_2 h_\beta \cosh\beta l + \beta a_1 h_\beta \sinh\beta l + \xi a_2 f_\xi \cos\xi l + a_1 \xi h_\beta \sin\xi l = 0 \tag{87}$$

$$a_1\left(\beta - h_\beta\right)\cosh\beta l + a_2\left(\beta - h_\beta\right)\sinh\beta l - a_1\frac{h_\beta}{f_\xi}\left(\xi - f_\xi\right)\cos\xi l + a_2\left(\xi - f_\xi\right)\sin\xi l = 0 \tag{88}$$

For the solutions a_1 and a_2 are non-zero, then there must have:

$$\begin{vmatrix} \beta h_\beta \sinh\beta l + \xi h_\beta \sin\xi l & \beta h_\beta \cosh\beta l + \xi f_\xi \cos\xi l \\ \left(\beta - h_\beta\right)\cosh\beta l - \dfrac{h_\beta}{f_\xi}\left(\xi - f_\xi\right)\cos\xi l & \left(\beta - h_\beta\right)\sinh\beta l + \left(\xi - f_\xi\right)\sin\xi l \end{vmatrix} = 0$$

or

$$\begin{aligned} &\left(\beta h_\beta \sinh\beta l + \xi h_\beta \sin\xi l\right)\left(\left(\beta - h_\beta\right)\sinh\beta l + \left(\xi - f_\xi\right)\sin\xi l\right) \\ &-\left(\beta h_\beta \cosh\beta l + \xi f_\xi \cos\xi l\right)\left(\left(\beta - h_\beta\right)\cosh\beta l - \frac{h_\beta}{f_\xi}\left(\xi - f_\xi\right)\cos\xi l\right) = 0 \end{aligned} \tag{89}$$

Equation (89) is the characteristic equation that determines natural frequencies, which is a nonlinear algebraic equation. Using the Newton – Raphson numerical method to solve the equation (89) with ω_j are the values to be determined, we will find the frequencies:

$$\omega_j \text{ with } j = 1, 2, ..., N_L \text{ so that } \omega_j < \omega_c$$

Mode shapes. From the formula (85) and (86) we deduce the coefficients corresponding to ω_j:

$$a_2^{(j)} = -a_1^{(j)} \frac{\left(\beta_j - h_\beta^{(j)}\right)\cosh\beta_j l - \dfrac{h_\beta^{(j)}}{f_\xi^{(j)}}\left(\xi_j - f_\xi^{(j)}\right)\cos\xi_j l}{\left(\beta_j - h_\beta^{(j)}\right)\sinh\beta_j l + \left(\xi_j - f_\xi^{(j)}\right)\sin\xi_j l} = -a_1^{(j)} g_j \tag{90}$$

$$a_3^{(j)} = -a_1^{(j)} \frac{h_\beta^{(j)}}{f_\xi^{(j)}}; \quad a_4^{(j)} = -a_2^{(j)} = a_1^{(j)} g_j$$

in which

$$h_\beta^{(j)} = h_\beta\left(\omega_j\right),\ f_\xi^{(j)} = f_\xi\left(\omega_j\right), \xi_j = \xi\left(\omega_j\right),\ \beta_j = \beta\left(\omega_j\right)$$

$$g_j = \frac{\left(\beta_j - h_\beta^{(j)}\right)\cosh\beta_j l - \dfrac{h_\beta^{(j)}}{f_\xi^{(j)}}\left(\xi_j - f_\xi^{(j)}\right)\cos\xi_j l}{\left(\beta_j - h_\beta^{(j)}\right)\sinh\beta_j l + \left(\xi_j - f_\xi^{(j)}\right)\sin\xi_j l}$$

Substitute the coefficients in (59) and (60) we get mode shapes

$$W_j(x) = a_1^{(j)}\left(\sinh\beta_j x + g_j \cosh\beta_j x - \left(h_\beta^{(j)} / f_\xi^{(j)}\right)\sin\xi_j x - g_j \cos\xi_j x\right) \tag{91}$$

$$\Phi_j(x) = a_1^{(j)}\left(h_\beta^{(j)} g_j \sinh\beta_j x + h_\beta^{(j)}\cosh\beta_j x + f_\xi g_j \sin\xi_j x - h_\beta^{(j)}\cos\xi_j x\right) \tag{92}$$

Case $\omega > \omega_c$

Natural frequency. Substituting (61), (62) into the boundary conditions (80) and (81), doing the same we get:

The characteristic equation to determine the natural frequencies

$$\begin{aligned}&\left(\eta f_\eta \sin\eta l - \xi f_\eta \sin\xi l\right)\left(\left(\eta - f_\eta\right)\sin\eta l - \left(\xi - f_\xi\right)\sin\xi l\right)\\&+\left(\eta f_\eta \cos\eta l - \xi f_\xi \cos\xi l\right)\left(\left(\eta - f_\eta\right)\cos\eta l - \frac{f_\eta}{f_\xi}\left(\xi - f_\xi\right)\cos\xi l\right) = 0\end{aligned} \tag{93}$$

Using numerical method to solve the Eq. (93) we will find the frequencies:

$$\omega_k \text{ with } k = 1, 2, ..., N_L \text{ stars so that } \omega_k > \omega_c$$

Mode shapes:

$$\mathrm{W}_k(x) = b_1^{(k)}\left(\sin\eta_k x + z_k \cos\eta_k x - \left(f_\eta^{(k)} / f_\xi^{(k)}\right)\sin\xi_k x - z_k \cos\xi x\right) \tag{94}$$

$$\Phi_k(x) = b_1^{(k)}\left(-z_k f_\eta^{(k)} \sin\eta_k x + f_\eta^{(k)}\cos\eta_k x + z_k f_\xi \sin\xi_k x - f_\eta^{(k)}\cos\xi x\right) \tag{95}$$

where $f_\eta^{(k)} = f_\eta\left(\omega_k\right),\ f_\xi^{(k)} = f_\xi\left(\omega_k\right), \xi_k = \xi\left(\omega_k\right),\ \beta_k = \beta\left(\omega_k\right)$

$$z_k = \frac{\left(\eta_k - f_\eta^{(k)}\right)\cos\eta_k l - \left(f_\eta^{(k)} / f_\xi^{(k)}\right)\left(\xi_k - f_\xi^{(k)}\right)\cos\xi_k l}{\left(\eta_k - f_\eta^{(k)}\right)\sin\eta_k l - \left(\xi_k - f_\xi^{(k)}\right)\sin\xi_k l}$$

Numerical results. The properties of the beam are: length l = 7.62 m, cross-section area A = 5.90 $\times$ 10^{-3} m^2, E = 2.14 $\times$ 10^{11} N/m^2, G = 8.18 $\times$ 10^{10} N/m^2, inertia moment of cross-sectional I = 4.58 $\times$ 10^{-5} m^4, mass of beam m = 350 kg, κ = 5/6.

The calculation results are compared with the EB beam in [20], the frequencies of it determined by analytical solution:

$$\omega = \frac{\lambda^2}{l^2}\sqrt{\frac{EI}{\rho A}} \tag{96}$$

where is λ the solution of the characteristic equation:

$$\cos\lambda\cosh\lambda + 1 = 0 \tag{97}$$

The calculation results are as shown in Table 2. On Fig.5 is the mode shapes of transverse displacement and cross-sectional rotation of CF PTM beam.

The results in Table 2 show that natural frequencies change rule of CF beam is similar to the case of SS beam. The first order natural frequency will deviate from the TM beam natural frequency the most (about 5.37% corresponding to $\varepsilon_0 = -3\times10^{-4}$).

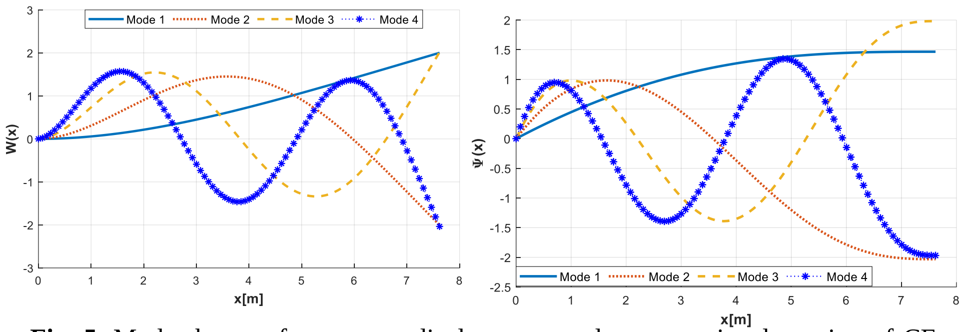


Fig. 5. Mode shapes of transverse displacement and cross-sectional rotation of CF PTM beam

Table 2. PTM beam natural frequencies (Hz)

	PTM (a) (Hz)			**TM** (b) (Hz)	**EB [20]** (Hz)	(a-b)/b (%)
	$\varepsilon_0 = -10^{-4}$	$\varepsilon_0 = -3.10^{-4}$	$\varepsilon_0 = 10^{-4}$	$\varepsilon_0 = 0$		
1	4.399	4.204	4.490	4.443	4.451	5.37
2	27.582	27.310	27.744	27.651	27.899	1.23
3	76.422	76.08775	76.665	76.507	78.119	0.55
4	147.324	146.9587	147.654	147.417	153.082	0.31
5	238.563	238.180	238.993	238.661	253.057	0.20
6	347.881	347.487	348.426	347.981	378.023	0.14

7	472.9906	472.5916	473.6632	473.0928	527.9831	0.11
8	611.6942	611.2933	612.5064	611.7969	702.9361	0.08
9	761.9721	761.5722	762.9342	762.0745	902.8825	0.07
10	922.0253	921.6863	922.7573	922.1267	1127.822	0.05

3.3 Clamped-Support PTM Beam

Boundary conditions of clamped – support (**CS**) beams

At $x = 0$, displacement and rotation are both 0:

$$w(0,t) = 0 \Rightarrow W(0) = 0; \quad \phi(0,t) = 0 \Rightarrow \Phi(0) = 0 \qquad (98)$$

At $x = l$, displacement is 0, moment is 0, similar to CF beam we have:

$$w(l,t) = 0 \Rightarrow W(l) = 0; \quad \frac{\partial \phi(l,t)}{\partial x} = 0 \Rightarrow \Phi'(l) = 0 \qquad (99)$$

Case $\omega < \omega_c$. Substituting the Eq.s (59), (60) into (98), (99), transform we get:

Natural frequency. Characteristic equation to determine frequencies:

$$\left(\sinh \beta l - \frac{h_\beta}{f_\xi}\sin \xi l\right)\left(\beta h_\beta \cosh \beta l + \xi f_\xi \cos \xi l\right) - \left(\cosh \beta l - \cos \xi l\right)\left(\beta h_\beta \sinh \beta l + h_\beta \xi \sin \xi l\right) = 0$$

(100)

Mode shapes.

$$W_j(x) = a_1^{(j)}\left(\sinh \beta_j x + g_j \cosh \beta_j x - \frac{h_\beta^{(j)}}{f_\xi^{(j)}}\sin \xi_j x - g_j \cos \xi_j x\right) \qquad (101)$$

$$\Phi_j(x) = a_1^{(j)}\left(h_\beta^{(j)} g_j \sinh \beta_j x + h_\beta^{(j)} \cosh \beta_j x + f_\xi g_j \sin \xi_j x - h_\beta^{(j)} \cos \xi_j x\right) \qquad (102)$$

where

$$h_\beta^{(j)} = h_\beta(\omega_j),\; f_\xi^{(j)} = f_\xi(\omega_j), \xi_j = \xi(\omega_j),\; \beta_j = \beta(\omega_j),\; g_j = \frac{\left(h_\beta^{(j)} / f_\xi^{(j)}\right)\sin \xi_j l - \sinh \beta_j l}{\cosh \beta_j l - \cos \xi_j l}$$

Case $\omega > \omega_c$. Substituting (61), (62) into (98) and (99), transform we get:

Natural frequency. Characteristic equation:

$$\left(\sin \eta l - \frac{f_\eta}{f_\xi}\sin \xi l\right)\left(-\eta f_\eta \cos \eta l + \xi f_\xi \cos \xi l\right) - \left(\cos \eta l - \cos \xi l\right)\left(-\eta f_\eta \sin \eta l + \xi f_\eta \sin \xi l\right) = 0$$

Mode shapes

$$W_k(x) = b_1^{(k)}\left(\sin \eta_k x + z_k \cos \eta_k x - \left(f_\eta^{(k)} / f_\xi^{(k)}\right)\sin \xi_k x - z_k \cos \xi x\right) \qquad (103)$$

$$\Phi_k(x) = b_1^{(k)}\left(-z_k f_\eta^{(k)} \sin \eta_k x + f_\eta^{(k)} \cos \eta_k x + z_k f_\xi \sin \xi_k x - f_\eta^{(k)} \cos \xi x\right) \qquad (104)$$

where

$$f_\eta^{(k)} = f_\eta\left(\omega_k\right),\ f_\xi^{(k)} = f_\xi\left(\omega_k\right), \xi_k = \xi\left(\omega_k\right),\ \beta_k = \beta\left(\omega_k\right), z_k = \frac{\left(f_\eta^{(k)} / f_\xi^{(k)}\right)\sin\xi_k l + \sin\eta_k l}{\cos\eta_k l - \cos\xi_k l}$$

Numerical results. The properties of the CS beam are similar to the CF beam. The calculation results are compared with the EB beam in [20], frequencies of it determined by analytical solution:

$$\omega = \frac{\lambda^2}{l^2}\sqrt{\frac{EI}{\rho A}} \tag{105}$$

where is λ the solution of the characteristic equation:

$$\tan\lambda - \tanh\lambda = 0 \tag{106}$$

The results presented in Table 3 show that the change of natural frequencies of PTM beam compared to TM beam and EB beam are similar to the above cases. The mode shapes show in Fig.7 are consistent with the boundary conditions of the CS beam. In Fig.8 shows the relative difference of frequencies of PTM beam versus TM beam. The difference increases as the prestress increases, the difference decreases rapidly as the frequency order increases.

Table 3. Natural frequencies of PTM CS beam (Hz)

	PTM beam (Hz)				**TM** (Hz)	**EB [20]** (Hz)	**(a-b)/b** (%)
	$\varepsilon_0 = -1.10^{-4}$	$\varepsilon_0 = -2.10^{-4}$ **(a)**	$\varepsilon_0 = 1.10^{-4}$	$\varepsilon_0 = 2.10^{-4}$	$\varepsilon_0 = 0$ **(b)**		
1	19.042	18.680	19,744	20.086	19.396	19,522	3.69
2	61.770	61.366	62,569	62,964	62.171	63.264	1.29
3	127.273	126.857	128,100	128.511	127.687	131,995	0.65
4	213.626	213.209	214.458	214,873	214.043	225,720	0.39
5	318.628	318.213	319,455	319,867	319.041	344.438	0.26
6	439.987	439.578	440,803	441.211	440.395	488.150	0.19
7	575.474	575.073	576,277	576,677	575.875	656.854	0.14
8	880.780	880.394	881,550	881,935	881.165	850.552	0.09
9	1047.12	1046.74	1047.87	1048.255	1047.50	1069.24	0.07
10	1220.66	1220.29	1221,40	1221.768	1221.03	1312.92	0.06

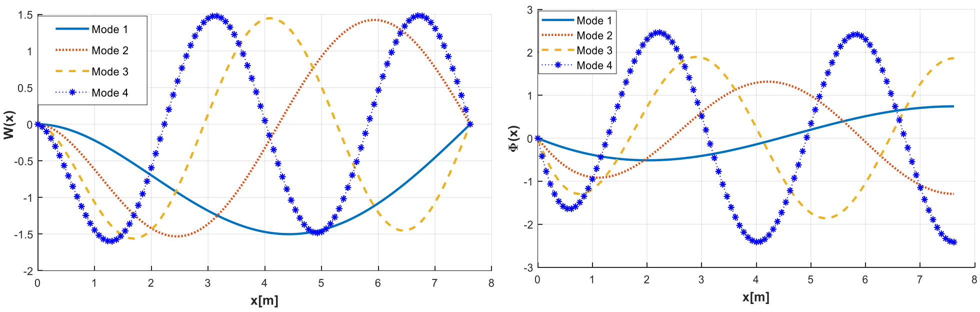


Fig. 6. Mode shapes of transverse displacement and cross-sectional rotation of clamped –support PTM beam

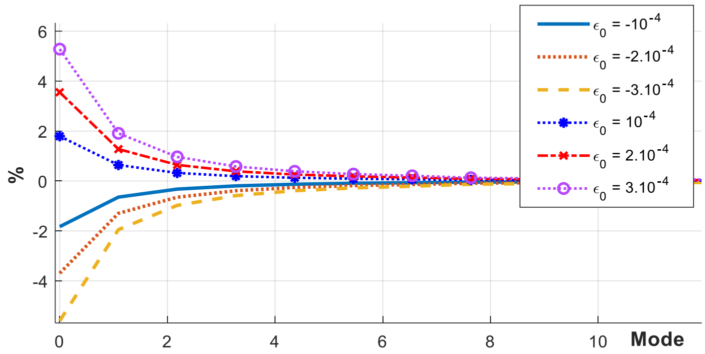


Fig .7. The relative difference of frequencies of PTM beam versus TM beam

4 Conclusions

This study is based on the vibration equations that has been established for the TM beam [6-9] to establish the vibration equations of PTM beam and general form of the mode functions by the MAM method, which the proof of dissociation of solutions is presented.

Using that general form of mode functions with specific boundary conditions, the study established characteristic equations to determine the natural frequencies and specific mode shapes for some PTM beam models such as SS beam, CF beam and CS beam. Other types of beams can also be determined in a similar. The article has performed numerical calculations for those beam models, the calculation results are compared with TM beams [6] or EB beams [20] showing:

Calculations give similar results for beams TM [6] when the prestress of PTM beams is zero ($\varepsilon_0 = 0$). The results also show the agreement with the EB beam [20]. The mode shapes presented in the figures show that they are suitable for the corresponding boundary conditions.

When the beams are pre-compressed ($\varepsilon_0 < 0$), the difference in frequencies of the PTM beam compared to the TM beam are increased as the prestress increases, in the direction the frequency of the PTM beam is smaller. When the beams are pre-tension ($\varepsilon_0 > 0$), this difference also increases, but in the direction the frequency of the PTM beam is larger than that of the TM beam. The difference will be greatest at first frequency and it is significant. This difference decreases rapidly as the order of the frequencies increases.

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